# Announcements

1) HW 2 up on Canvas:
4 webwork questions,
pdf under "Assignments"
webwork due Friday,
rest due Tuesday next
week.

2) Ford Day tomorrow!
Presentation 12:15 Kochoff

# Linear Equations

( Section 2.3 )

Recall definition

A differential equation of the form

$$\boxed{a_1(x)\frac{dy}{dx} + a_2(x)y = a_3(x)}$$

( 1st order )

## Motivating Example:

$$\cos(x)\, y + \frac{dy}{dx}\sin(x) = x\ln(x)$$

not a separable equation.

We can integrate $x\ln(x)$ using integration by parts. What about the left-hand side?

The left hand side is actually the derivative of a product:

$$\frac{d}{dx}\left(y\sin(x)\right)$$

$$= \underbrace{\frac{dy}{dx}\sin(x) + y\cos(x)}_{\text{product rule}}$$

Integrate both sides
with respect to $x$!

$$\int x\ln(x)\,dx$$

$$= \int\left(\frac{dy}{dx}\sin(x) + y\cos(x)\right)dx$$

$$= \int \frac{d}{dx}\left(y\sin(x)\right)dx$$

$$= y\sin(x) + C$$

Integrate $x\ln(x)$

by parts:

$$u = \ln(x) \quad v = \frac{x^2}{2}$$

$$du = \frac{1}{x}dx \quad dv = x\,dx$$

$$\int u\,dv = uv - \int v\,du$$

$$= \frac{x^2}{2}\ln(x) - \int \frac{x^{\cancel{2}}}{2}\cdot\frac{1}{\cancel{x}}dx$$

$$= \frac{x^2}{2}\ln(x) - \frac{x^2}{4}$$

Equating the two sides,

$$\frac{x^2}{2}\ln(x) - \frac{x^2}{4} = y\sin(x) + C \text{, so}$$

$$\boxed{y = \frac{\frac{x^2}{2}\ln(x) - \frac{x^2}{4} - C}{\sin(x)}}$$

Example 2: Solve

$$\frac{dy}{dx} - y - e^{3x} = 0$$

Rewrite as

$$\frac{dy}{dx} - y = e^{3x}.$$

We can integrate $e^{3x}$.

Is $\frac{dy}{dx} - y$ the derivative of a product?

$\frac{dy}{dx} - y$ is not

the derivative of

$a(x)y$ because then

$a(x) = 1$ and $a'(x) = -1$

by the chain rule, this

is impossible. We want

to make $\frac{dy}{dx} - y$

into the derivative of

a product!

Use an Integrating Factor:

Multiply

$$\frac{dy}{dx} - y \quad \text{by} \quad a(x)$$

such that

$$a(x)\frac{dy}{dx} - a(x)y$$

$$= \frac{d}{dx}\left(a(x)y\right).$$

Can choose $a(x) = e^{-x}$.

If $a(x) = e^{-x}$,

$$\frac{d}{dx}\left(e^{-x} y\right)$$

$$= e^{-x}\frac{dy}{dx} + y\left(-e^{-x}\right)$$

$$= e^{-x}\frac{dy}{dx} - e^{-x} y$$

$$= a(x)\frac{dy}{dx} - a(x) y$$ ✓

Multiply both sides of the equation by $e^{-x}$:

$$e^{-x}\frac{dy}{dx} - e^{-x}y = e^{-x}e^{3x}$$
$$= e^{2x}$$

Integrate both sides with respect to $x$.

$$\int \left(e^{-x}\frac{dy}{dx} - e^{-x}y\right)dx$$

$$= \int \frac{d}{dx}\left(e^{-x}y\right)dx$$

$$= e^{-x}y + C$$

and

$$\int e^{2x}dx = \frac{e^{2x}}{2}.$$ Then

$$e^{-x}y + C = \frac{e^{2x}}{2}.$$

Solving for $y$,

$$y = \frac{\frac{e^{2x}}{2} + C}{e^{-x}}$$

$$= \left(\frac{e^{2x}}{2} + C\right)e^{x}$$

$$= \boxed{\frac{e^{3x}}{2} + Ce^{x}}$$