

Announcements

- 1) HW 2 up on Canvas:
4 webwork questions,
pdf under "Assignments"
webwork due Friday,
rest due Tuesday next
week.
- 2) Ford Day tomorrow!
Presentation 12:15 Kochhoff

Linear Equations

(Section 2,3)

Recall definition

A differential equation
of the form

$$a_1(x) \frac{dy}{dx} + a_2(x)y = a_3(x)$$

(1st order)

Motivating Example.

$$\cos(x) y + \frac{dy}{dx} \sin(x) = x \ln(x)$$

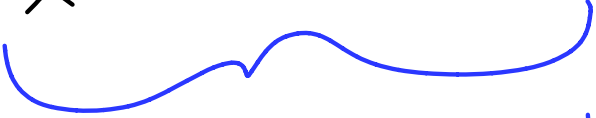
not a separable equation.

We can integrate $x \ln(x)$ using integration by parts. What about the left-hand side?

The left hand side
is actually the
derivative of a
product:

$$\frac{d}{dx} (y \sin(x))$$

$$= \frac{dy}{dx} \sin(x) + y \cos(x)$$


product rule

Integrate both sides
with respect to x :

$$\int x \ln(x) dx$$

$$= \int \left(\frac{dy}{dx} \sin(x) + y \cos(x) \right) dx$$

$$= \int \frac{d}{dx} (y \sin(x)) dx$$

$$= y \sin(x) + C$$

Integrate $x \ln(x)$

by parts:

$$U = \ln(x) \quad v = \frac{x^2}{2}$$

$$du = \frac{1}{x} dx \quad dv = x dx$$

$$\int u dv = uv - \int v du$$

$$= \frac{x^2}{2} \ln(x) - \int \frac{x^2}{2} \cdot \frac{1}{x} dx$$

$$= \frac{x^2}{2} \ln(x) - \frac{x^2}{4}$$

Equating the two
sides,

$$\frac{x^2}{2} \ln(x) - \frac{x^2}{4} =$$

$y \sin(x) + C$, so

$$y = \frac{\frac{x^2}{2} \ln(x) - \frac{x^2}{4} - C}{\sin(x)}$$

Example 2: Solve

$$\frac{dy}{dx} - y - e^{3x} = 0$$

Rewrite as

$$\frac{dy}{dx} - y = e^{3x}.$$

We can integrate e^{3x} .

Is $\frac{dy}{dx} - y$ the derivative

of a product?

$\frac{dy}{dx} - y$ is not

the derivative of

$a(x)y$ because then
 $a(x) = 1$ and $a'(x) = -1$

by the chain rule, this
is impossible. We want

to **make** $\frac{dy}{dx} - y$

into the derivative of
a product!

Use an Integrating Factor:

Multiply

$$\frac{dy}{dx} - y \text{ by } a(x)$$

such that

$$a(x) \frac{dy}{dx} - a(x)y$$

$$= \frac{d}{dx} (a(x)y).$$

(can choose $a(x) = e^{-x}$).

$$\text{If } a(x) = e^{-x},$$

$$\frac{d}{dx} (e^{-x} y)$$

$$= e^{-x} \frac{dy}{dx} + y (-e^{-x})$$

$$= e^{-x} \frac{dy}{dx} - e^{-x} y$$

$$= a(x) \frac{dy}{dx} - a(x) y \quad \checkmark$$

Multiply both sides
of the equation by
 e^{-x} :

$$e^{-x} \frac{dy}{dx} - e^{-x} y = e^{-x} e^{3x}$$
$$= e^{2x}$$

Integrate both sides
with respect to x .

$$\int (e^{-x} \frac{dy}{dx} - e^{-x} y) dx$$

$$= \int \frac{d}{dx} (e^{-x} y) dx$$

$$= e^{-x} y + C$$

and

$$\int e^{2x} dx = \frac{e^{2x}}{2}. \text{ Then}$$

$$e^{-x} y + C = \frac{e^{2x}}{2}.$$

Solving for y,

$$y = \frac{\frac{e^{2x}}{2} + C}{e^{-x}}$$

$$= \left(\frac{e^{2x}}{2} + C \right) e^x$$

$$= \boxed{\frac{e^{3x}}{2} + Ce^x}$$